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THE OPTIMAL DESIGN OF NEAR-PERIODIC STRUCTURES TO MINIMIZE VIBRATION TRANSMISSION AND STRESS LEVELS

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This work is concerned with the optimal design of a near-periodic beam system to minimize vibration transmission and also maximum stress levels. Both narrow-band and wide-band excitation are considered, and two sets of design parameters are investigated; namely, the individual bay lengths and the individual bay damping values. It is found that very significant reductions in the selected objective functions can be achieved with relatively minor design changes. In the case of the system bay lengths, it is found that a near-optimum design can usually be found by employing a "bound search" algorithm, which obviates the need for a full optimization routine. Furthermore, it is shown that vibration transmission cannot readily be reduced by changing the damping distribution in the system (while maintaining the same amount of "total" damping), although significant reductions in the maximum stress levels can be obtained through this means. The findings of the work have application to more general engineering structures which are of near-periodic construction.

1. INTRODUCTION

An engineering structure is said to be of "periodic" construction if a basic structural unit is repeated in a regular pattern. A beam which rests on regularly spaced supports is one example of a one-dimensional periodic structure, while an orthogonally stiffened cylinder is an example of a two-dimensional periodic structure. It has long been known that perfectly periodic structures have very distinctive vibration properties, in the sense that "pass bands" and "stop bands" arise: these are frequency bands over which elastic wave motion respectively can and cannot propagate through the structure [1, 2]. If the excitation frequency lies within a stop band, then the structural response tends to be localized to the immediate vicinity of the excitation source. Conversely, if the excitation frequency lies within a pass band, then strong vibration transmission can occur, and it is generally the case that the resonant frequencies of the structure lie within the pass bands.

Much recent work has been performed concerning the effect of random disorder on a nominally periodic structure (see, for example, [3–5]). It has been found that disorder can lead to localization of the response even for excitation which lies within a pass band, and this reduces the propensity of the structure to transmit vibration. This raises the possibility of deliberately *designing* disorder into a structure in order to reduce vibration transmission, and this possibility was briefly investigated in reference [6] for a one-dimensional periodic waveguide which was embedded in an otherwise infinite homogeneous system. In this regard, it should be noted that much previous work has been performed in the area of optimal structural design, and it is now fairly routine to run an optimization algorithm in conjunction with finite element analysis software to produce a design which is optimized

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for static or dynamic loading [7]. However, relatively little work has been directed specifically at optimal structural design to reduce vibration transmission and stress levels, where the concern tends to be with wide-band excitation at relatively high frequencies. In this case the nature of the excitation is such that a fine mesh finite element model is needed to capture the short wavelength of the high frequency structural deformation, and the computational demands of such a model render standard optimization procedures impracticable—in fact, a single response analysis of a system with fixed design parameters can be computationally impracticable at high frequencies. The specific issue of design to minimize vibration transmission has recently been addressed by Keane [8]. For the case of a planar framework, a receptance structural analysis method was coupled to a genetic algorithm optimization scheme to yield an efficient (and feasible) computational scheme. It was shown that significant vibration reductions could be obtained across a fairly wide frequency band, and this suggests that similar reductions might be achievable for near-periodic systems.

In the present study, the work reported in reference [6] is extended for an embedded waveguide to the case of a finite near-periodic beam system which more closely resembles the type of optimization problem likely to occur in engineering practice. The beam is taken to have N bays, and the design parameters are taken to be the individual bay lengths, and individual bay damping. Both single frequency and band-limited excitation are considered, and two objective functions are investigated: (i) the response in a bay which is distant from the applied loading (minimization of vibration transmission), and (ii) the maximum response in the structure (minimization of maximum stress levels). In each case the optimal configuration is found by employing a quasi-Newton algorithm, and the physical features of the resulting design are discussed in order to suggest general design guidelines.

2. ANALYTICAL MODEL OF THE NEAR-PERIODIC BEAM

2.1. CALCULATION OF THE FORCED RESPONSE

A schematic of an N-bay near-periodic beam structure is shown in Figure 1. The structure is subjected to dynamic loading, and the aim of the present work is to find the optimal design which will minimize a prescribed measure of the vibration response. No matter what type of optimization algorithm is employed, this type of study requires repeated computation of the system dynamic response as the design parameters are varied, and it is therefore important to employ an efficient analysis procedure. In the present work the h-p version of the finite element method (FEM) is employed with this approach, the structure is modelled as an assembly of elements that have both nodal and internal degrees of freedom. Each element has two nodes and the nodal degrees of freedom consist of the beam displacement and slope. The internal degrees of freedom are generalized co-ordinates, which are associated with an hierarchy of shape functions that contribute only to the internal displacement field of the element. The internal shape functions used here are the K-orthogonal Legendre polynomials of order four onwards. Full details of

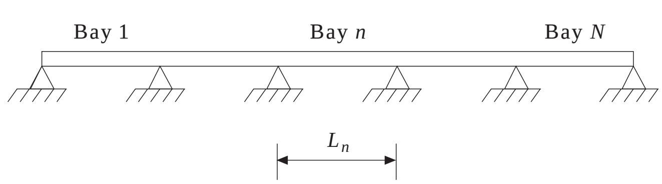

Figure 1. A schematic of a simply supported N-bay periodic beam.

the present modelling approach are given in reference [9], where the method was used to study the effects of asymmetry on the dynamics of a periodic beam system.

For harmonic excitation of frequency ω, the equations of motion of the complete beam structure can be written in the form

$$[-\omega^2\mathbf{M} + (1 + \mathrm{i}\eta)\mathbf{K}]\mathbf{q} = \mathbf{F}, \tag{1}$$

where $\mathbf{M}$ and $\mathbf{K}$ are the global mass and stiffness matrices (assembled from the individual element matrices taking into account the presence of any mass or spring attachments and allowing for constraints), $\mathbf{q}$ contains the system degrees of freedom, $\mathbf{F}$ is the vector of applied forces, and η is the loss factor, which in the first part of the present study is taken to be uniform throughout the structure. For non-uniform damping, η can be taken inside $\mathbf{K}$ and assigned a different value for each element. In equation (1) and all subsequent equations, the time dependency $e^{\mathrm{i}\omega t}$ is implicitly assumed for harmonic variables.

Equation (1) can readily be solved to yield the system response $\mathbf{q}$. In the present work it is convenient to use the time averaged kinetic and strain energies of each of the N bays as a measure of the response—for the nth bay these quantities can be written as T_n and U_n say, where

$$T_n = \frac{\omega^2}{4}\mathbf{q}_n^{*\mathrm{T}}\mathbf{M}_n\,\mathbf{q}_n, \qquad U_n = \tfrac{1}{4}\mathbf{q}_n^{*\mathrm{T}}\mathbf{K}_n\,\mathbf{q}_n. \tag{2, 3}$$

Here $\mathbf{M}_n$ and $\mathbf{K}_n$ are the mass and stiffness matrices of the nth bay, and $\mathbf{q}_n$ is the degree of freedom vector for this bay.

Many of the physical features of the forced response of a near-periodic structure can be explained in terms of the free vibration behaviour of the associated perfectly periodic structure. The following section outlines how the present finite element modelling approach can be used to study the pass bands and stop bands exhibited by a perfect periodic structure.

2.2. PERIODIC STRUCTURE ANALYSIS

The finite element method described in section 2.1 can be applied to a *single bay* of a perfectly periodic structure to yield an equation of motion in the form

$$\mathbf{D}\mathbf{q} = \mathbf{F}, \qquad \mathbf{D} = -\omega^2\mathbf{M} + (1 + \mathrm{i}\eta)\mathbf{K}, \tag{4, 5}$$

where the matrix $\mathbf{D}$ is referred to as the dynamic stiffness matrix, and $\mathbf{M}$, $\mathbf{K}$, $\mathbf{q}$ and $\mathbf{F}$ now relate to the particular bay under consideration. In order to study wave motion through the periodic system, it is convenient to partition $\mathbf{D}$, $\mathbf{q}$ and $\mathbf{F}$ as follows:

$$\mathbf{D} = \begin{pmatrix} \mathbf{D}_{LL} & \mathbf{D}_{LI} & \mathbf{D}_{LR} \\ \mathbf{D}_{IL} & \mathbf{D}_{II} & \mathbf{D}_{IR} \\ \mathbf{D}_{RL} & \mathbf{D}_{RI} & \mathbf{D}_{RR} \end{pmatrix}, \qquad \mathbf{q} = \begin{pmatrix} \mathbf{q}_L \\ \mathbf{q}_I \\ \mathbf{q}_R \end{pmatrix}, \qquad \mathbf{F} = \begin{pmatrix} \mathbf{F}_L \\ \mathbf{0} \\ \mathbf{F}_R \end{pmatrix}, \tag{6–8}$$

where L relates to the degrees of freedom at the leftmost node, R relates to those at the rightmost node and I relates to the remaining "internal" degrees of freedom. Equations (4–8) can be used to derive the following transfer matrix relation between the displacements and forces at the left- and right-hand nodes:

$$\mathbf{T}\begin{pmatrix} \mathbf{q}_L \\ \mathbf{F}_L \end{pmatrix} = \begin{pmatrix} \mathbf{q}_R \\ -\mathbf{F}_R \end{pmatrix}, \qquad \mathbf{T} = \begin{pmatrix} \mathbf{D}_{LL} - \mathbf{D}_{LI}\,\mathbf{D}_{II}^{-1}\,\mathbf{D}_{IL} & \mathbf{D}_{LR} - \mathbf{D}_{LI}\,\mathbf{D}_{II}^{-1}\,\mathbf{D}_{IR} \\ \mathbf{D}_{RL} - \mathbf{D}_{RI}\,\mathbf{D}_{II}^{-1}\,\mathbf{D}_{IL} & \mathbf{D}_{RR} - \mathbf{D}_{RI}\,\mathbf{D}_{II}^{-1}\,\mathbf{D}_{IR} \end{pmatrix}. \tag{9, 10}$$

Equation (9) can now be used to analyse wave motion through the periodic system: such motion is governed by Bloch's Theorem [10], which states that $(\mathbf{q}_L \;\; \mathbf{F}_L)^{\mathrm{T}} = \exp(-\mathrm{i}\varepsilon - \delta)\,(\mathbf{q}_R \;\; -\mathbf{F}_R)^{\mathrm{T}}$, where ε and δ are known, respectively, as the phase and attenuation constants. A pass band is defined as a frequency band over which $\delta = 0$, so that wave motion can propagate through the structure without attenuation. It follows from equation (9) that

$$(\mathbf{T} - \mathbf{I}\,\mathrm{e}^{-\mathrm{i}\varepsilon-\delta})\begin{pmatrix}\mathbf{q}_L\\ \mathbf{F}_L\end{pmatrix} = \begin{pmatrix}\mathbf{0}\\ \mathbf{0}\end{pmatrix}, \tag{11}$$

so that ε and δ can be computed from the eigenvalues of $\mathbf{T}$, thus enabling the pass bands and stop bands to be identified.

2.3. OPTIMIZATION PROCEDURE

Equations (1)–(3) enable the forced response of a general beam system to be calculated for any prescribed set of system properties. The aim of the present analysis is to compute the *optimal* set of system properties for a prescribed design objective, and in order to achieve this, equations (1)–(3) are evaluated repeatedly as part of an optimization algorithm. As an example, it might be required to minimize the kinetic energy of bay N by changing the various bay lengths. In this case, equations (1)–(3) provide the route via which the objective function (the kinetic energy in bay N) is related to the design parameters (the bay lengths), and the optimization algorithm must adjust the design parameters so as to minimize the objective function. The optimization process has been performed here by using the NAG library routine E04UCF [11], which employs a quasi-Newton algorithm. This type of algorithm locates a minimum in the objective function, although there is no indication as to whether this minimum is the global minimum or a less optimal local minimum. The probability of locating the global minimum can be increased significantly by repeated application of the NAG routine using random starts; i.e., by prescribing random initial values of the initial design parameters. Numerical investigations have led to the use of 30 random starts in the present work.

2.4. OBJECTIVE FUNCTIONS AND FORCING

The examples presented in the following section each concern an N-bay beam system which is excited by a point force located within bay 1. To remove any sensitivity of the response to the precise position of the force, the response has been averaged over 11 equally spaced locations within the bay—this gives an approximation to a "rain-on-the-roof" forcing of bay 1. The frequency content of the applied loading is an important design driver, and the following four cases are considered: (i) single frequency loading within the second pass band of the ordered structure, (ii) band-limited loading lying within the second pass band; (iii) band-limited loading covering the whole of the second pass band; (iv) band-limiting loading which covers the second stop band and the second pass band.

Two objective functions are considered in determining the optimal design, corresponding to the minimization of vibration transmission and the minimization of maximum stress levels: (i) the kinetic energy of the final bay T_N; and (ii) the maximum bay strain energy U_n occurring in the structure. In case (ii), the bay strain energy is taken as a convenient indicator of the system stress levels to avoid the need for a detailed stress recovery.

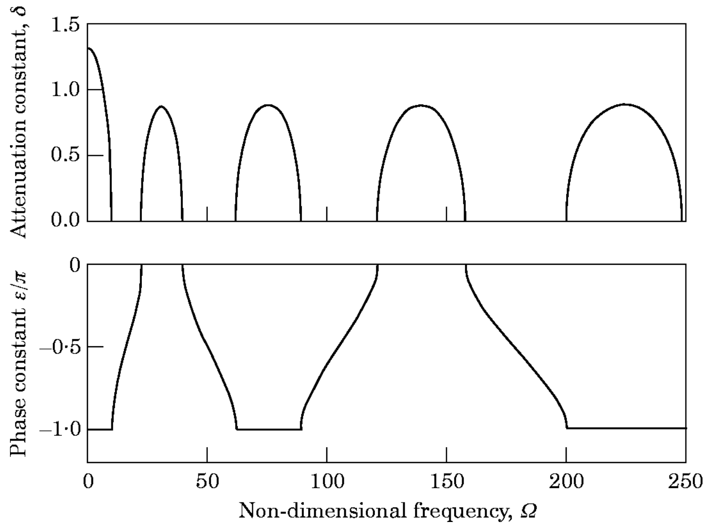

Figure 2. Propagation constants for a simply supported beam.

3. NUMERICAL RESULTS

3.1. SYSTEM I: BEAM ON MULTIPLE SIMPLE SUPPORTS WITH VARIABLE BAY LENGTHS

The foregoing analysis has been applied to a beam of flexural rigidity EI, mass per unit length m and loss factor $\eta = 0{\cdot}015$, which rests on $N + 1$ simple supports, thus giving an N-bay near-periodic system. The design parameters are taken to be the bay lengths (i.e., the separation of the simple supports), and the design is constrained so that the length L_n of any bay lies within the range $0{\cdot}9L_r \leqslant L_n \leqslant 1{\cdot}1L_r$, where L_r is a reference length. A non-dimensional frequency Ω is introduced such that $\Omega = \omega L_r^2\sqrt{(m/EI)}$, and the non-dimensional kinetic and strain energies of a bay are defined as $T_n' = T_n\,(EI/L_r^3\,|F|^2)$ and $U_n' = U_n\,(EI/L_r^3\,|F|^2)$, where F is the applied point load. For reference, the propagation constants for a periodic system in which all the bay lengths are equal to L_r are shown in Figure 2—the present study is focused on excitation frequencies which lie in the range $23 \leqslant \Omega \leqslant 61$, which covers the second stop band and the second pass band of the periodic system. In the second pass band, the beam deformation has a wavelength which is approximately equal to L_r: the current permitted change in bay length $0.9L_r \leqslant L_n \leqslant 1{\cdot}1L_r$ is therefore equivalent to around 20% of the vibrational wavelength, and this implies that significant modifications to the system's behaviour should be achievable.

3.1.1. *Design for minimum vibration transmission*

In this case, the objective function is taken to be the kinetic energy in bay N, so that the aim is to minimize the vibration transmitted along the structure. As mentioned in section 2.4, four types of loading are considered: (i) single frequency loading with $\Omega = 50$, which lies within the second pass band of the ordered structure; (ii) narrow band-limited loading with $46 \leqslant \Omega \leqslant 54$; (iii) band-limited loading with $40 \leqslant \Omega \leqslant 60$, which covers the whole of the second pass band; (iv) band-limited loading with $23 \leqslant \Omega \leqslant 61$, which covers the whole of the second stop band and the second pass band.

Results for the optimal design under single frequency loading are shown in Table 1; in all cases it was found that the bay lengths were placed against either the upper bound ($U = 1{\cdot}1L_r$) or the lower bound ($L = 0{\cdot}9L_r$), and significant reductions in the energy level of bay N were achieved. In this regard it should be noted that the dB reduction quoted

TABLE 1

The optimal design of a 1-D *beam structure to minimize energy transmission at* $\Omega = 50$. *The design parameter is bay lengths, the "Original energy" is the non-dimensional kinetic energy in bay N of the periodic structure, and the "Final energy" is the non-dimensional kinetic energy in bay N of the optimized structure*

No. of bays, N	Optimal pattern energy	Original energy	Final energy	Reduction (dB)
4	UULU	0·276E1	0·804E − 3	35·4
5	ULULU	0·609E−1	0·179E − 3	25·3
8	UULULULU	0·674E0	0·613E − 5	50·4
9	ULULULULU	0·564E−1	0·135E − 5	46·2
10	UULULULULU	0·424E0	0·523E − 6	59·0
11	ULULULULULU	0·535E−1	0·117E − 6	56·6
12	UULULULULULU	0·289E0	0·461E − 7	68·0
13	ULULULULULULU	0·502E−1	0·101E − 7	67·0
16	UULULULULULULULU	0·154E0	0·346E − 9	86·5
17	ULULULULULULULULU	0·431E−1	0·761E − 10	87·5

in Table 1 is defined as $-10 \log (T_N / T_{Nr})$, where T_{Nr} is the kinetic energy in the final bay of the ordered system. The optimal designs shown in Table 1 all tend to consist of a bi-periodic structure, in which the basic unit consists of two bays in the configuration LU. The pass bands and stop bands for this configuration are shown in Figure 3, and, furthermore, T_N for the optimal 12-bay system is shown in Figure 4 over the extended frequency range $0 \leqslant \Omega \leqslant 250$. By comparing Figures 2 and 3 it is clear why the selected design is optimal—the new bi-periodic system has a stop band centred on the specified excitation frequency $\Omega = 50$. As would be expected, it can be seen from Figure 4 that the improvement in the response at the specified frequency $\Omega = 50$ is accompanied by a worsening of the response at some other frequencies.

Results for the optimal design under narrow band-limited excitation over the range $46 \leqslant \Omega \leqslant 54$ are shown in Table 2. In some cases two results are shown for the optimized

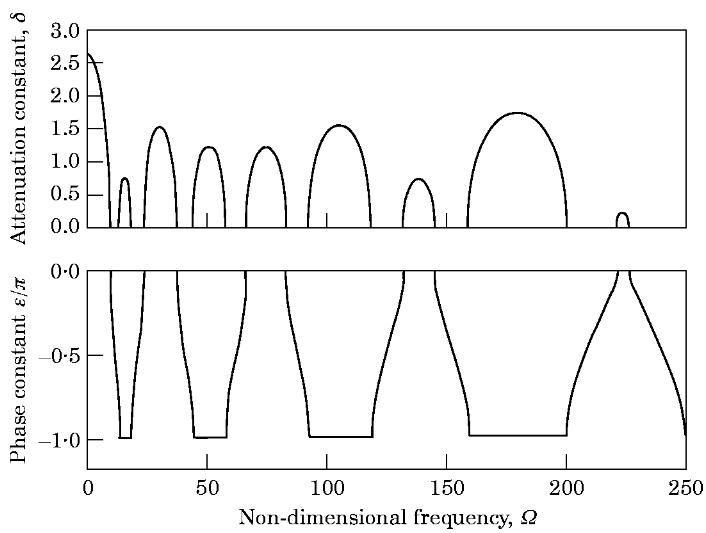

Figure 3. Propagation constants for a bi-periodic simply supported beam.

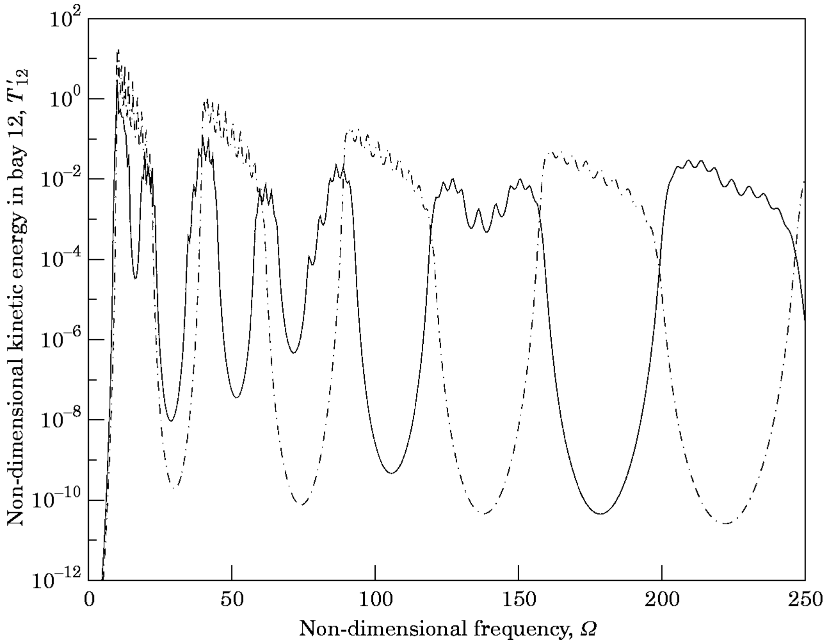

Figure 4. The frequency response of a 12-bay beam: kinetic energy in bay 12, optimized for $\Omega = 50$, obtained by adjusting bay length. —·—·—, The original structure; ——, the optimized structure, "UULULULULULU".

"Final Energy": in such cases the first result has been obtained by "forcing" each bay length on to either the upper (U) or lower (L) bound, while the second result has been obtained by using the NAG optimization routine. If only one result is shown, then the two methods yield the same optimal design. The "bound" result is easily obtained by computing the response under each possible combination of U and L bay lengths—this

Table 2

The optimal design of a 1-D *beam structure to minimize energy transmission for* $46 \leqslant \Omega \leqslant 54$. *The design parameter is the bay lengths, the "Original energy" is the non-dimensional kinetic energy in bay N of the periodic structure, and the "Final energy" is the non-dimensional kinetic energy in bay N of the optimized structure*

No. of bays, N	Optimal pattern	Original energy	Final energy	Reduction (dB)
4	UULU	0·403E0	0·178E − 2	23·5
5	ULULU	0·492E0	0·271E − 2	22·6
6	UULULU	0·216E0	0·230E − 3	29·7
7	ULULULU	0·250E0	0·301E − 4	39·2
8	UULULULU	0·257E0	0·297E − 4	39·4
	NAG	0·257E0	0·295E − 4	39·4
9	ULULULULU	0·162E0	0·348E − 5	46·7
10	ULULLULULU	0·178E0	0·346E − 5	47·1
11	ULULULULULU	0·157E0	0·418E − 5	45·7
12	ULULULLULULU	0·127E0	0·395E − 6	55·1
	NAG	0·127E0	0·391E − 6	55·1
13	ULULULULULULU	0·129E0	0·521E − 7	63·9
14	ULULULLULULULU	0·105E0	0·464E − 7	63·5

TABLE 3

The optimal design of a 1-D *beam structure to minimize energy transmission for* $40 \leqslant \Omega \leqslant 60$. *The design parameter is the bay lengths, the "Original energy" is the non-dimensional kinetic energy in bay N of the periodic structure, and the "Final energy" is the non-dimensional kinetic energy in bay N of the optimized structure*

No. of bays, N	Optimal pattern	Original energy	Final energy	Reduction (dB)
4	ULLU	0·670E0	0·103E − 1	18·1
5	ULLLU	0·631E0	0·735E − 2	19·3
	NAG	0·631E0	0·711E − 2	19·5
6	UULLLU	0·384E0	0·221E − 2	22·4
7	ULLUULU	0·463E0	0·171E − 2	24·3
8	UULLLLLU	0·430E0	0·966E − 3	26·5
	NAG	0·430E0	0·914E − 3	26·7
9	UUULLLLLU	0·444E0	0·341E − 3	31·1
10	UUUULLLLLU	0·449E0	0·192E − 3	33·7
	NAG	0·449E0	0·189E − 3	33·8
11	ULLUUUULLLU	0·291E0	0·821E − 4	35·5
12	UUULUULLLLLU	0·201E0	0·352E − 4	37·6
13	ULUUUUULLLLLU	0·199E0	0·153E − 4	41·1

requires 2^N response calculations, which normally takes much less CPU time than the NAG optimization routine. It is clear from Table 2 that the additional improvement in the response yielded by the full optimization routine is minimal for this case. The optimal structural configuration is similar to that obtained under single frequency excitation. This is unsurprising as the frequency range $46 \leqslant \Omega \leqslant 56$ is still within the stop band generated by the new bi-periodic structure. However, for even bay numbers of $N \geqslant 10$, two adjacent lower bounds are set near the centre of the structure. This, in effect, causes the structure to be made up of two bi-periodic structures of design "UL", connected back to back.

Results for the optimal design under band-limited excitation over the range $40 \leqslant \Omega \leqslant 60$ are shown in Table 3. The response curve for the 12-bay system is shown in Figure 5, where it is clear that a significantly reduced response is achieved over the specified frequency range: as would be expected, an increase in the response occurs at other frequencies. It is interesting to note that most of the optimal designs shown in Table 3 lack symmetry—however, it follows from the principle of reciprocity that a design that minimizes vibration transmission from left to right will also minimize transmission from right to left. It should therefore be possible to "reverse" the designs without changing the transmitted vibration levels. This hypothesis is tested in Figure 6 for a 12-bay structure: shown in the figure is the energy distribution for the optimal design UUULUULLLLLU and for the reversed design ULLLLLUULUUU. Although the detailed distribution of energy varies between the two designs, the energy levels achieved in bay 12 are identical, as expected.

Results for the optimal design under wide-band excitation $23 \leqslant \Omega \leqslant 61$ are shown in Table 4, and the response curve for the 12-bay optimized system is shown in Figure 7. The form of optimal design achieved is similar to that obtained for the narrower excitation band $40 \leqslant \Omega \leqslant 60$, although there are detailed differences between the two sets of results. In each case there is a tendency for a group of lower bound bays (L) to occur in the mid region of the structure, and a group of upper bound bays (U) to occur at either end. This creates an "impedance mismatch" between the two sets of bays, which promotes wave reflection and thus reduces vibration transmission along the structure. By comparing Tables 1–4, it is clear that the amount of reduction in vibration transmission that can be achieved reduces as the bandwidth of the excitation is increased.

3.1.2. *Design for minimum "maximum" strain energy*

In this case the strain energy U_n of each bay is computed and the objective function is taken to be the maximum value of U_n. As in the previous section, the four frequency ranges $\Omega = 50$, $46 \leqslant \Omega \leqslant 54$, $40 \leqslant \Omega \leqslant 60$ and $23 \leqslant \Omega \leqslant 61$ are considered, and the optimal structures obtained for $N = 9$–12 are shown in Table 5.

Considering the single frequency results ($\Omega = 50$) shown in Table 5, it is clear that a large dB reduction is achieved only for those systems which have an even number of bays; furthermore, the optimal energy obtained has the same value (0·0297) in all cases. This can be explained by noting that for an odd number of bays the frequency $\Omega = 50$ lies near to an anti-resonance of the ordered structure, whereas a resonance is excited for an even number of bays—this feature is illustrated in Figure 8 for the 12-bay structure. The repeated occurrence of the optimal energy 0·0297 arises from the fact that the initial bay pattern ULLLUUU occurs in all designs. It has been found that this pattern causes a vibration reduction of over 20 dB from bay 1 to bay 8, so that the response in bay 1 (the maximum response) is insensitive to the nature of the structure from bay 8 onwards. This is illustrated in Figure 9, in which the energy distribution in the optimized 12-bay system is shown.

The results shown in Table 5 for the three types of band-limited excitation display a number of general features: (i) in contrast to the single frequency case, the dB reduction achievable is not sensitive to having either an even or an odd number of bays; (ii) in

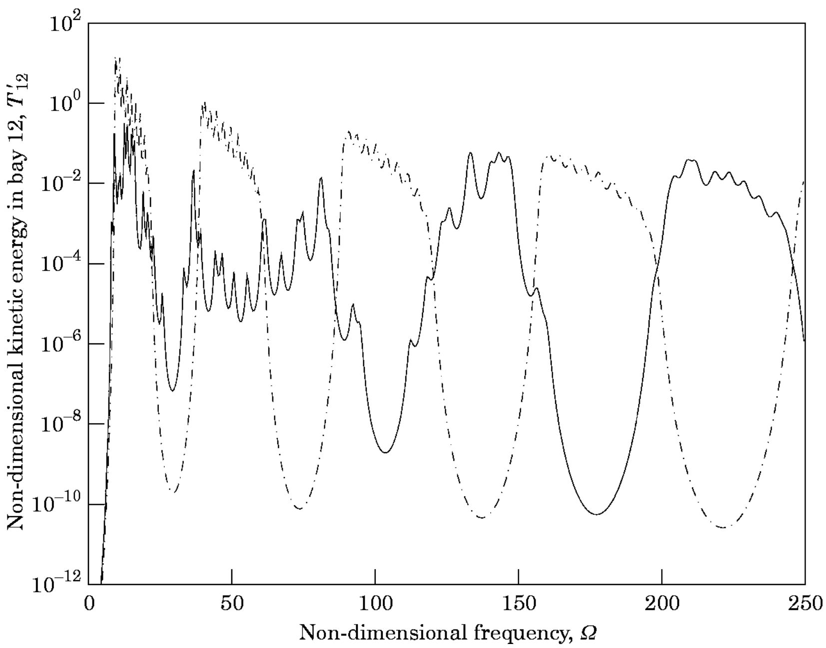

Figure 5. The frequency response of a 12-bay beam: kinetic energy in bay 12, optimized for $40 \leqslant \Omega \leqslant 60$, obtained by adjusting bay length. —·—·—, The original structure; ——, the optimized structure, "UUULUULLLLLU".

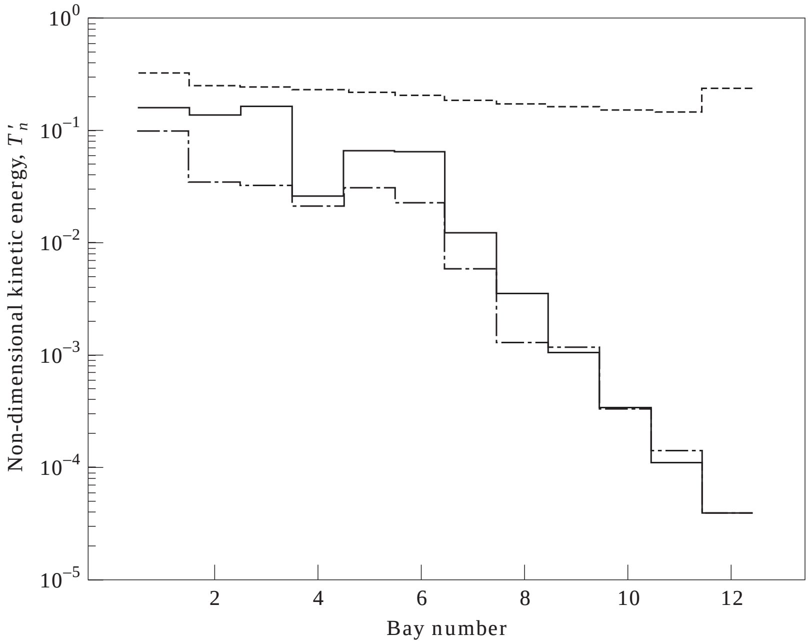

Figure 6. The distribution of kinetic energy within a 12-bay beam, showing reciprocity. Also shown is the kinetic energy within the original beam. —·—·—·—, The structure "ULLLLLUULUUU"; ——, the structure "UUULUULLLLLU"; – – – – –, the original structure.

TABLE 4

The optimal design of a 1-D *beam structure to minimize energy transmission for* $23 \leqslant \Omega \leqslant 61$. *The design parameter is the bay lengths, the "Original energy" is the non-dimensional kinetic energy in bay N of the periodic structure, and the "Final energy" is the non-dimensional kinetic energy in bay N of the optimized structure*

No. of bays, N	Optimal pattern	Original energy	Final energy	Reduction (dB)
4	LLUU	0·536E0	0·581E − 1	9·7
	NAG	0·536E0	0·383E − 1	11·5
5	LLLUU	0·340E0	0·180E − 1	12·8
	NAG	0·340E0	0·138E − 1	13·9
6	LLLLUU	0·494E0	0·648E − 2	18·8
7	LLLULUU	0·183E0	0·246E − 2	18·7
8	LLLLLUUU	0·175E0	0·180E − 2	19·9
9	ULLLLLLUU	0·139E0	0·904E − 3	21·9
10	UUULLLLLUU	0·105E0	0·277E − 3	25·8
11	UUULLLLLLUU	0·105E0	0·776E − 4	31·3
12	UUULLLLLLLUU	0·166E0	0·526E − 4	35·0
13	UUULLLLLLULUU	0·973E−1	0·282E − 4	35·4
14	UUUULLLLLLULUU	0·581E−1	0·122E − 4	36·8

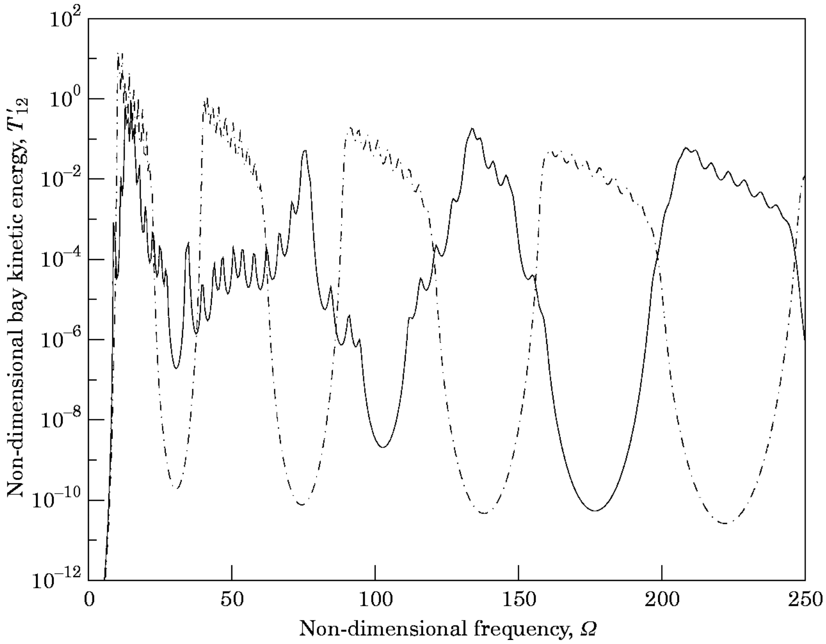

Figure 7. The frequency response of a 12-bay beam; kinetic energy in bay 12, optimized for $23 \leqslant \Omega \leqslant 61$, obtained by adjusting bay length. —·—·—·, The original structure; and ——, the optimized structure, "UUULLLLLLLUU".

agreement with the single frequency case, the optimal energy is relatively insensitive to the number of bays in the system; (iii) the optimal energy yielded by the "bound search" method becomes significantly greater than that yielded by the full optimization procedure as the excitation bandwidth is increased. Point (i) arises simply because the excitation bandwidth always encompasses at least one resonant mode of the system, and thus there is no equivalent of the "anti-resonant" case that can arise for $\Omega = 50$. Point (ii) is due to the fact that the maximum energy arises in or near to the excited bay, and this energy is little affected by the more remote parts of the system. Point (iii) is important from a computational point of view, since for wide-band excitation the simple "bound search" algorithm cannot be relied upon to produce a near-optimal result. This is illustrated in Figure 10 for the case of a 12-bay system subjected to wide-band excitation $23 \leqslant \Omega \leqslant 61$; the "bound search" optimal design is UUUUUUUUULUU, whereas the design yielded by the full optimization corresponds to bay lengths (as a multiple of L_r) of 0·900; 0·988; 0·979: 1·007; 0·986; 1·019; 0·986; 1·026; 0·987; 1·058; 1·021; 0·948. As shown in Table 5, the maximum stress energy is more than 3 dB lower for the fully optimized design.

The performance of the present optimization strategy can be assessed by considering the range of results yielded by the 30 random starts employed in the search procedure. For the 12-bay system with $23 \leqslant \Omega \leqslant 61$, it was found that the 30 random starts yielded a range of minimum values between $1{\cdot}55 \times 10^{-1}$ and $7{\cdot}95 \times 10^{-2}$ for the non-dimensional strain energy; the dB reduction associated with those designs ranged from 3·1 dB to 6·0 dB. It cannot be guaranteed (although it is likely) that the present approach has yielded the global minimum. Further confidence could be achieved by employing more random starts or by using an alternative optimisation procedure such as a genetic algorithm [8]. The present results can be viewed at worst as a lower bound on the achievable vibration reduction, and at best the globally optimal reduction.

3.2. SYSTEM II: BEAM ON MULTIPLE SIMPLE SUPPORTS WITH VARIABLE DAMPING DISTRIBUTION

A uniform simply supported beam with the same flexural rigidity and mass per unit length as the beam in System I is analyzed in this section. The loss factor in each bay, η_n, is now the design parameter. The initial structure has a uniform loss factor of $\eta_n = 0{\cdot}05$, which gives an average modal overlap factor of $M = 0{\cdot}125\,N$ for the second pass band

TABLE 5

The optimal design of 1-D beam structure to minimize "maximum" strain energy. The design parameter is the bay lengths. N' is the bay in which the optimal minimum "maximum" non-dimensional strain energy occurs, the "Original energy" is the initial "maximum" non-dimensional strain energy of the periodic structure, and the "Final energy" is the non-dimensional strain energy in bay N' of the optimized structure

No. of bays, N	Optimal pattern	Original energy	Final energy	Bay no. N'	Reduction (dB)
$\Omega = 50$					
9	ULLLUUUUL	0·667E−1	0·297E − 1	1	3·5
	NAG	0·667E−1	0·296E − 1	1	3·5
10	ULLLUUUULU	0·540E0	0·297E − 1	1	12·6
	NAG	0·540E0	0·296E − 1	1	12·6
11	ULLLUUUULUU	0·691E−1	0·297E − 1	1	3·7
	NAG	0·691E−1	0·296E − 1	1	3·7
12	ULLLUUUULULU	0·404E0	0·297E − 1	1	11·3
	NAG	0·404E0	0·296E − 1	1	11·4
$46 \leqslant \Omega \leqslant 54$					
9	ULLLLLULL	0·181E0	0·340E − 1	1	7·3
	NAG	0·181E0	0·331E − 1	1	7·4
10	ULULLLLULL	0·207E0	0·344E − 1	1	7·8
	NAG	0·207E0	0·329E − 1	1	8·0
11	ULLLLLULLLU	0·193E0	0·343E − 1	1	7·5
	NAG	0·193E0	0·328E − 1	1	7·7
12	ULLLLLLLLULL	0·161E0	0·341E − 1	1	6·7
	NAG	0·161E0	0·328E − 1	1	6·9
$40 \leqslant \Omega \leqslant 50$					
9	UUUUULULU	0·486E0	0·710E − 1	1	8·4
	NAG	0·486E0	0·449E − 1	1–2	10·3
10	ULULULULLL	0·606E0	0·643E − 1	1	9·7
	NAG	0·606E0	0·451E − 1	1–2	11·3
11	ULULULULLUU	0·456E0	0·682E − 1	1	8·3
	NAG	0·456E0	0·425E − 1	1–2	10·3
12	UUUUUUUUUUUL	0·332E0	0·550E − 1	2	7·8
	NAG	0·332E0	0·412E − 1	1–2	9·2
$23 \leqslant \Omega \leqslant 61$					
9	LLLLLLLLL	0·234E0	0·203E0	1	0·6
	NAG	0·234E0	0·979E − 1	1	3·8
10	LLLLLLLLLL	0·200E0	0·178E0	1	0·5
	NAG	0·200E0	0·951E − 1	1–2	3·2
11	UUUUUUUUULL	0·198E0	0·193E0	1	0·1
	NAG	0·198E0	0·910E − 1	1–2	3·4
12	UUUUUUUUULUU	0·314E0	0·182E0	1	2·4
	NAG	0·314E0	0·795E − 1	1–2	6·0

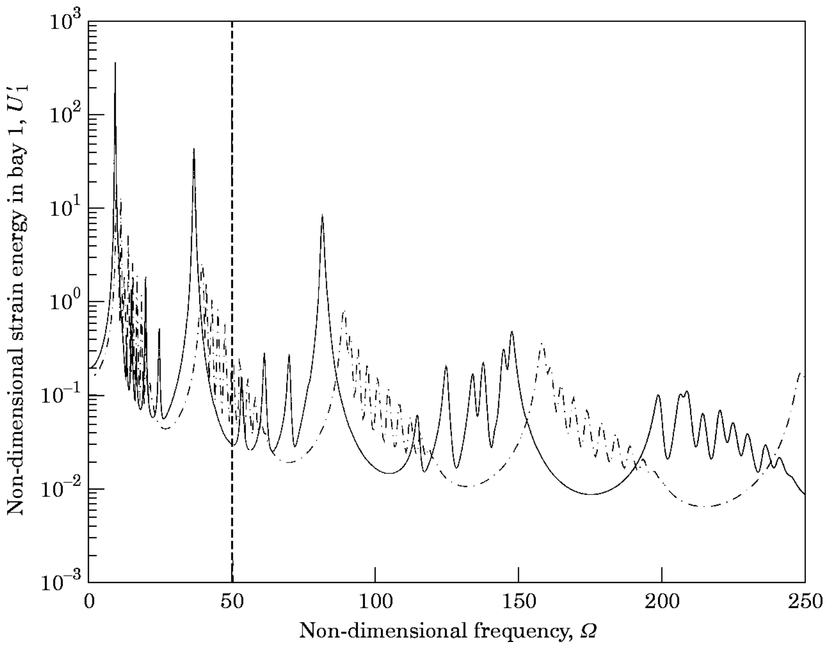

Figure 8. The frequency response of a 12-bay beam: strain energy in bay 1, optimized for $\Omega = 50$, obtained by adjusting bay length. —·—·—·, The original structure; ——, the optimized structure, "ULLLUUUULULU".

of an N-bay system (based on the fact that the band has a width of $\Delta\Omega = 20$, a centre frequency of $\Omega_c = 50$, and N modes lie in the band so that $M = \Omega_c\,\eta(N/\Delta\Omega) = 0{\cdot}125\,N$). The modal overlap factor gives an indication of the smoothness of the frequency response

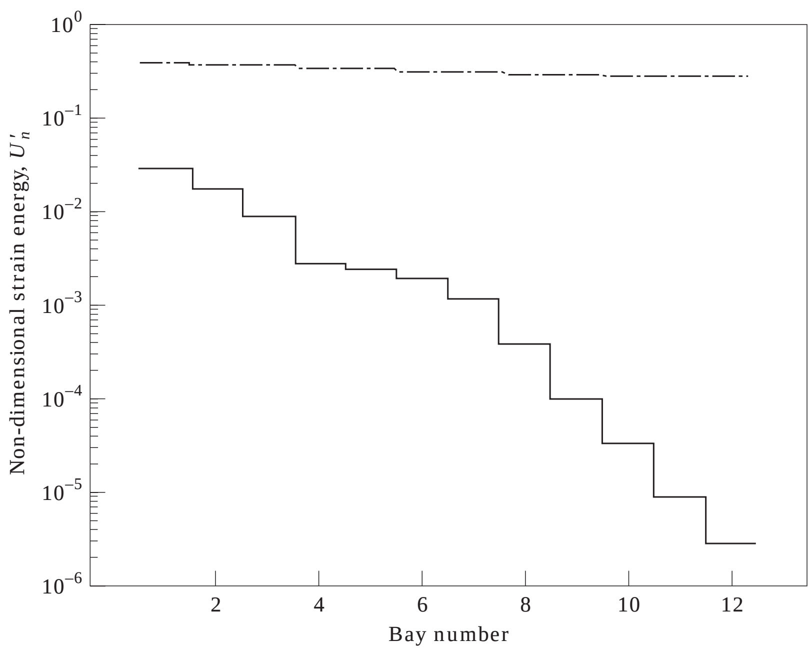

Figure 9. The distribution of strain energy within a 12-bay beam, optimized for $\Omega = 50$, obtained by adjusting bay lengths. —·—·—·—, The original structure; ——, the optimized structure, "ULLLUUUULULU".

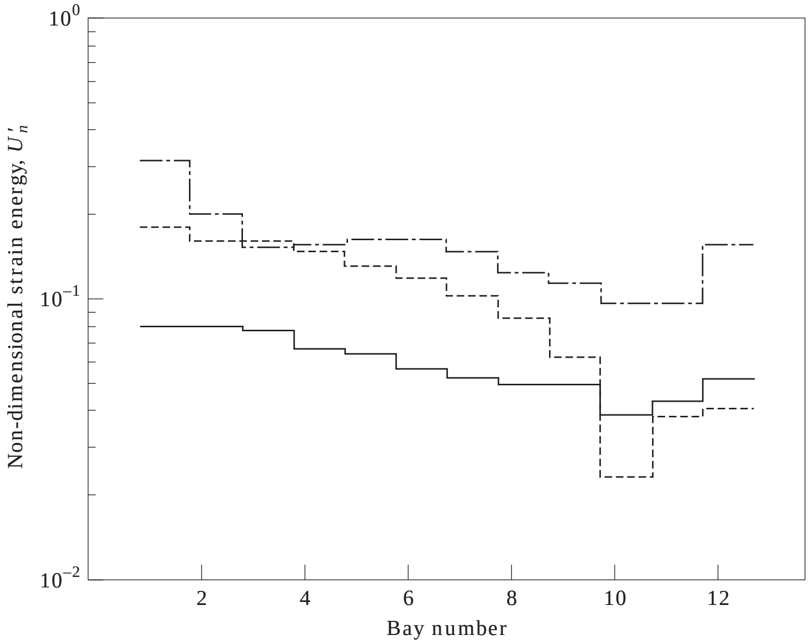

Figure 10. The distribution of strain energy within a 12-bay beam, optimized for $23 \leqslant \Omega \leqslant 61$. —·—·—·, The original structure; – – – – –, the "bound" obtained structure, "UUUUUUUUULUU"; ———, the "NAG" obtained structure "0·900, 0·988, 0·979, 1·007, 0·986, 1·019, 0·986, 1·026, 0·987, 1·058, 1·021, 0·948" (multiples of L_r).

function: if $M < 1$ the modes appear as distinct sharp peaks, whereas for $M > 1$ the resonant peaks merge, and the response function has a much smoother appearance. For periodic structures the modal density, and hence the modal overlap factor, can vary significantly over a pass band, with the greatest values occurring near the pass band edges. The average value $M = 0{\cdot}125\,N$ is used here to give an indication of the degree of modal overlap, and it is recognised that individual pairs of modes near to the band edges could, in principle, deviate from this result. However, for the range of examples considered here, this effect has not been found to be significant and M gives a good physical measure of the modal overlap.

In the present study the bay loss factors η_n are constrained such that

$$\sum_{n=1}^{N} \eta_n = 0{\cdot}05\,N, \tag{12}$$

which means that the "total" damping in the structure is held constant. Physically, the damping might be introduced by bonding a damping layer to the structure, in which case equation (12) corresponds to the use of a fixed total mass of damping material.

3.2.1. *Design for minimum vibration transmission*

Results for the optimized response of an N-bay system subjected to a single frequency excitation with $\Omega = 50$ are shown in Table 6: as in section 3.1.1 the objective function was taken to be the kinetic energy in bay N, and the NAG routine E04UCF was used to perform the optimization. It is immediately obvious from the results shown in Table 6 that the distribution of damping treatment is a very weak design parameter, in the sense that little reduction in the level of vibration transmitted can be effected. In fact, it is found that

TABLE 6

The optimal design of a 1-D *beam structure to minimize energy transmission at* $\Omega = 50$. *The design parameter is bay damping, the "Original energy" is the non-dimensional kinetic energy in bay N of the periodic structure, and the "Final energy" is the non-dimensional kinetic energy in bay N of the optimized structure*

No. of bays, N	Original modal overlap factor	Original energy	Final energy	Reduction (dB)
4	0·5	0·236E 0	0·232E 0	0·07
5	0.625	0·443E−1	0·427E − 1	0·16
6	0·75	0·947E−1	0·927E − 1	0·09
7	0·875	0·322E−1	0·312E − 1	0·14
8	1	0·463E−1	0·453E − 1	0·10
9	1·125	0·221E−1	0·214E − 1	0·14
10	1·25	0·250E−1	0·244E − 1	0·11
11	1·375	0·145E−1	0·141E − 1	0·12
12	1·5	0·141E−1	0·138E − 1	0·09

the kinetic energy of bay N is almost independent of the damping distribution, regardless of the value of the modal overlap factor or whether the system is resonant or non-resonant at the excitation frequency $\Omega = 50$. In order to explain this general insensitivity, it is necessary to consider three distinct cases: (i) a low modal overlap system excited off-resonance; (ii) a low modal overlap system excited at resonance; and (iii) a high modal overlap system. In case (i) it is obvious that the distribution of damping is of no significance, since the non-resonant response is independent of damping. In case (ii) it

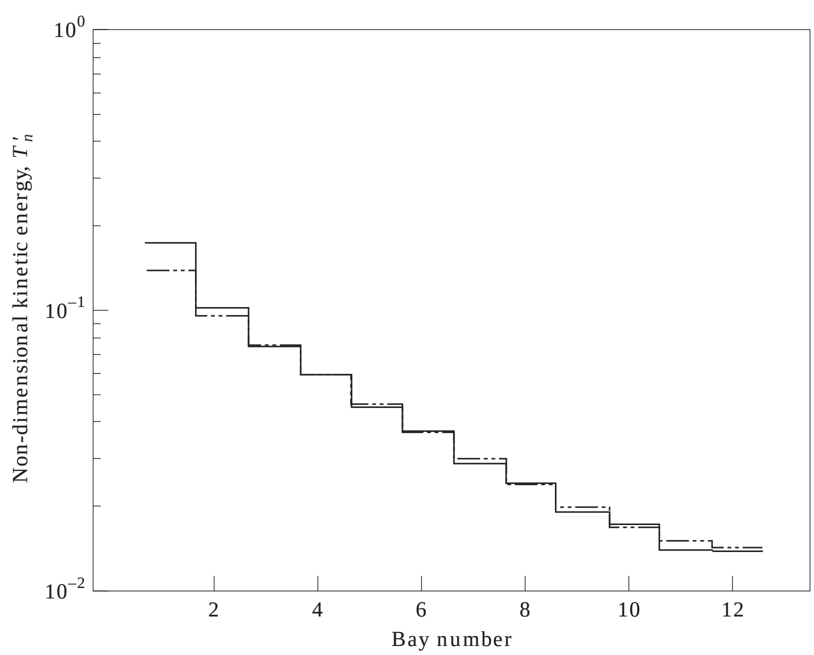

Figure 11. The distribution of kinetic energy within a 12-bay beam, optimized for $\Omega = 50$, obtained by adjusting bay damping. —·—·—·—, The original structure; ——, the optimized structure, $\eta_1 = \eta_{12} = 0{\cdot}00$, $\eta_2 = \eta_{11} = 0{\cdot}10$, $\eta_3 = \eta_{10} = 0{\cdot}44 \times 10^{-1}$, $\eta_4 = \eta_9 = 0{\cdot}54 \times 10^{-1}$, $\eta_5 = \eta_8 = 0{\cdot}49 \times 10^{-1}$, $\eta_6 = \eta_7 = 0{\cdot}50 \times 10^{-1}$.

TABLE 7

The optimal design of a 1-D *beam structure to minimize "maximum" strain energy at* $\Omega = 50$. *The design parameter is bay damping*, N' *is the bay in which the optimal minimum "maximum" non-dimensional strain energy occurs*, *the "Original energy" is the initial "maximum" non-dimensional strain energy of the periodic structure*, *and the "Final energy" is the non-dimensional strain energy in bay* N' *of the optimized structure*

No. of bays, N	Original energy	Final energy	Bay no., N'	Reduction (dB)
4	0·314E0	0·262E0	1–2	0·8
5	0·640E−1	0·493E−1	1–2	1·1
6	0·184E0	0·121E0	1–3	1·8
7	0·732E−1	0·423E−1	1–2	2·4
8	0·140E0	0·711E−1	1–2–3	2·9
9	0·814E−1	0·361E−1	1–2	3·5
10	0·121E0	0·479E−1	1–2	4·0
11	0·877E−1	0·306E−1	1–2	4·6
12	0·112E0	0·351E−1	1–2–3	5·0

should be recalled that the present concern is with a periodic structure having N bays. The generalized loss factor associated with a particular mode can be expressed in the form

$$\eta_{eff} = \sum_{n=1}^{N} \eta_n \int_{\text{bay } n} m\phi^2(x)\, \mathrm{d}x, \tag{13}$$

where $\phi(x)$ is the mode shape (scaled to unit generalized mass) and m is the mass per unit length of the beam. For a periodic structure, the modes are generally non-localized, and the integral that appears in equation (13) yields approximately the same result for each bay—given that the modes are scaled to unit generalized mass, the result is $(1/N)$. Equation (13) thus yields

$$\eta_{eff} \approx \sum_{n=1}^{N} \eta_n \frac{1}{N} = 0{\cdot}05, \tag{14}$$

so that the modal damping (and hence the system response) is independent of the spatial distribution of the damping. In case (iii), high modal overlap, it is less obvious that the transmitted energy will be independent of the damping distribution. However, it can be noted that a one-dimensional periodic system with high modal overlap has a relatively high attenuation constant δ arising from the presence of damping [12]. This implies that there is a significant decay in the amplitude of an elastic wave as it passes along the system—the amplitude decays by factor of $\exp(-\delta)$ over each bay; the total decay along the system can be written approximately as $\exp(-\Sigma_n \delta_n)$, where δ_n is the attenuation constant for bay n. The attenuation constant δ varies approximately linearly with the loss factor η [12], which implies that $\Sigma_n \delta_n \propto \Sigma_n \eta_n = 0{\cdot}05\, N$ and hence the total attenuation along the system is independent of the detailed distribution of damping. This type of behaviour is shown in Figure 11 for the case of a 12-bay system. It should be noted that the present argument is based on the assumption that the system is non-reverberant, so that the "direct field" response estimate $\exp(-\Sigma_n \delta_n)$ truly represents the response in bay N. This reasoning will not apply if a significant amount of energy is reflected from the right-hand end of the system, as might occur if all the damping is concentrated in the first few bays of the system.

Such a damping arrangement would tend to increase, rather than decrease, vibration transmission, and an example of this type of behaviour is given in the following section.

It follows from the above arguments that the levels of vibration transmitted along a periodic structure cannot significantly be reduced by modifying the spatial distribution of damping, and this is borne out by the results shown in Table 6. Although not detailed here, similar results have also been obtained for band-limited excitation. This behaviour will generally *not* occur for a disordered system—the modes of a disordered system tend to be localized, and it can be anticipated that damping will be employed to greatest effect in regions of high modal displacement.

3.2.2. *Design for minimum "maximum" strain energy*

As in section 3.1.2, the objective function is in this case taken to be the maximum bay strain energy U_n occurring in the system. Results for the optimized maximum energy are shown in Table 7 for the case of the single frequency excitation with $\Omega = 50$. It is clear that little reduction in the *maximum* energy can be obtained when the number of bays is small; this is because the system has a low modal overlap ($M = 0{\cdot}125\,N$) and, as explained in the previous section, neither the resonant nor the non-resonant response is sensitive to the damping distribution in this case. As the modal overlap increases, the single frequency force ($\Omega = 50$) begins to excite more than one mode (due to increasing modal bandwidth), and the distribution of damping has more effect on the objective function. The frequency response function of the optimized 12-bay system is shown in Figure 12, and the energy distribution at $\Omega = 50$ is shown in Figure 13; in this case the optimal damping consists of $\eta_1 = 0{\cdot}5229$, $\eta_2 = 0{\cdot}0241$ and $\eta_3 = 0{\cdot}0529$, with zero damping in all the other bays. It can be seen from Figure 13 that the optimal damping arrangement has led to a near uniform distribution of strain energy across the system, so that the reduction in the maximum strain energy (bay 1) is accompanied by an

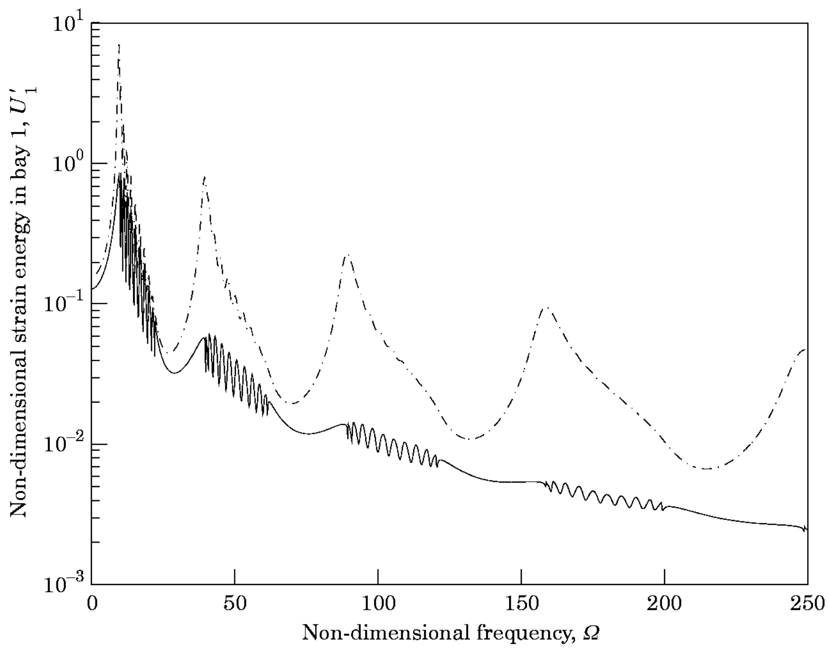

Figure 12. The frequency response of a 12-bay beam: strain energy in bay 1, optimized for $\Omega = 50$, obtained by adjusting bay damping. —·—·—·, The original structure; ——, the optimized structure, $\eta_1 = 0{\cdot}5229$, $\eta_2 = 0{\cdot}2413 \times 10^{-1}$, $\eta_3 = 0{\cdot}5293 \times 10^{-1}$, $\eta_4 = \eta_5 = \eta_6 = \eta_7 = \eta_8 = \eta_9 = \eta_{10} = \eta_{11} = \eta_{12} = 0{\cdot}0$.

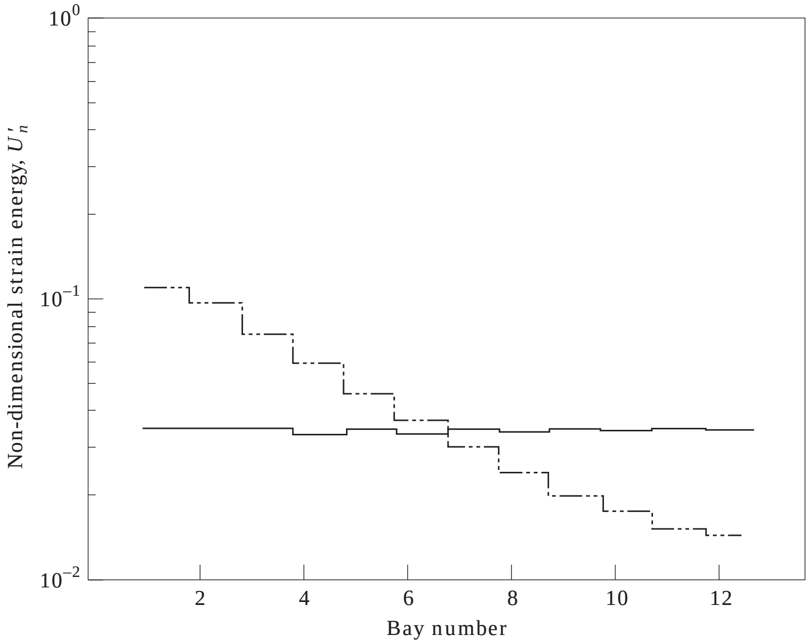

Figure 13. The distribution of strain energy within a 12-bay beam, optimized for $\Omega = 50$, obtained by adjusting bay damping. —·—·—·, The original structure; ——, the optimized structure, $\eta_1 = 0{\cdot}5229$, $\eta_2 = 0{\cdot}2413 \times 10^{-1}$, $\eta_3 = 0{\cdot}5293 \times 10^{-1}$, $\eta_4 = \eta_5 = \eta_6 = \eta_7 = \eta_8 = \eta_9 = \eta_{10} = \eta_{11} = \eta_{12} = 0{\cdot}0$.

increase in vibration transmission (bay 12). As discussed in the previous section, the present damping arrangement invalidates the argument that the transmitted energy should be independent of the damping distribution, since the response in bays 4–12 is highly reverberant.

It can be noted that the damping model employed in the present work is based on the use of a viscoelastic loss factor η. The bay value $\eta_1 = 0{\cdot}5229$ is extremely high, although not beyond quoted results for certain plastics and rubbers [13]; the practical aspects of achieving such a value would need to be considered in a design situation, and such

TABLE 8

The optimal design of a 1-D *beam structure to minimize "maximum" strain energy for* $40 \leqslant \Omega \leqslant 60$. *The design parameter is bay damping. N' is the bay in which the optimal minimum "maximum" non-dimensional strain energy occurs, the "Original energy" is the initial "maximum" non-dimensional strain energy of the periodic structure, and the "Final energy" is the non-dimensional strain energy in bay N' of the optimized structure*

No. of bays, N	Original energy	Final energy	Bay no., N'	Reduction (dB)
4	0·201E0	0·155E0	1–4	1·1
5	0·200E0	0·110E0	1–2–5	2·6
6	0·178E0	0·859E − 1	1–2–3–6	3·2
7	0·186E0	0·698E − 1	1–3–7	4·3
8	0·184E0	0·581E − 1	1–4–5–8	5·0
9	0·188E0	0·485E − 1	1–3	5·9
10	0·189E0	0·424E − 1	1–10	6·5
11	0·182E0	0·378E − 1	1–2	6·8
12	0·174E0	0·323E − 1	1–2–3	7·3

considerations could place additional bounds on the permissible values of η employed in the optimization procedure.

From the frequency–response curve shown in Figure 12, it would appear that the optimal damping arrangement has decreased the modal overlap of the system, since the individual resonant peaks are much more clearly visible for the optimized system. However, equation (13) and the subsequent discussion implies that the modal damping factors, and hence the modal overlap, should be *independent* of the damping distribution, which is in apparent contradiction with the results shown in Figure 12. This contradiction can be resolved by noting that the damping distribution produces a *coupling* damping factor between modes i and j of the form

$$\eta_{ij} = \sum_{n=1}^{N} \eta_n \int_{\text{bay } n} m\phi_i(x)\phi_j(x)\,\mathrm{d}x, \tag{15}$$

where $\phi_i(x)$ and $\phi_j(x)$ are the relevant mode shapes of the *undamped* system. If the damping is uniformly distributed, then equation (15) yields $\eta_{ij} = 0$ by virtue of the orthogonality of the mode shapes. If the damping distribution is highly non-uniform, then η_{ij} may have an appreciable value, and for high modal overlap this coupling term can have a significant effect on the system response, leading to the type of behaviour shown in Figure 12. To give an alternative, wave-based, explanation for the results shown in Figure 12, the visible resonant peaks are associated with wave phase closure over the undamped section of the structure.

Results for the optimized maximum energy for the case of wide-band excitation with $40 \leqslant \Omega \leqslant 60$ are shown in Table 8. Generally, greater reductions in the maximum energy are achieved than the single frequency case of $\Omega = 50$, and again it was found that the optimal damping was distributed near to the loaded bay to give a fairly uniform distribution of energy down the system. For the 12-bay system the optimal damping arrangement was found to be $\eta_1 = 0{\cdot}504$, $\eta_2 = 0{\cdot}0945$ and $\eta_3 = 0{\cdot}00145$, with zero damping in all other bays.

4. CONCLUSIONS

The present work has considered the optimal design of a near-periodic beam system to minimize vibration transmission and also maximum stress levels. Two sets of design parameters have been considered; namely, the individual bay lengths and the individual bay damping values. The main conclusions drawn from this work are summarized below.

(1) *Vibration transmission with variable bay length*. In this case the optimal design involves placing the structural design parameters (the bay lengths) on the permissible bounds, and this means that a simple design search routine can be used in preference to a full optimization algorithm. As would be expected, the obtainable reduction in vibration transmission decreases with increasing excitation bandwidth and increases with increasing system size.

(2) *Vibration transmission with variable bay damping*. It has been found that vibration transmission cannot substantially be reduced by changing the way in which damping is distributed through the system. A physical explanation of this result is presented in section 3.2.1 for the separate cases of low and high modal overlap.

(3) *Maximum stress levels with variable bay lengths*. In this case significant reductions can be achieved, although the fully optimal configuration must normally be found by employing a full optimization algorithm rather than a bound search approach. Nonetheless, in most cases the bound search approach produces a near-optimal structure.

In the case of single frequency excitation, the achievable reduction in maximum stress levels is relatively low if the original design displays an anti-resonance at the excitation frequency. For band-limited excitation, the achievable reduction normally decreases with increasing bandwidth, and is rather insensitive to the system size.

(4) *Maximum stress levels with variable bay damping*. In this case the maximum stress levels are reduced by concentrating the damping near to the excited bay. This produces a near uniform distribution of energy throughout the system, which has the side effect of increasing vibration transmission.

Clearly, the present study could be extended to consider a combined transmission/stress objective function, and also simultaneous use of the damping and bay lengths design parameters. The main purpose of the results reported here is to indicate that much improved wide-band vibration performance can be achieved in near-periodic structures by performing relatively minor design changes. This provides motivation for the study of more complex systems than the 1-D beam example considered here.

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